Maths for Computing
Tutorial 1

1. Think of some sentences that are neither questions nor instructions, but still not propositions.
Solution: One example is “This sentence is true”.

Consider the below box:

> 1) At least one of the statements in this box is false.
>
> 2) X is not going to top M4C.

In the above box, 1) is true because if it is false, then it will imply that no statement in the box is false which is a contradiction. But if 1) is true, then 2) has to be false. So, X will top the exam. How are we able to prove that X will top the M4C without knowing much about X? What’s the catch? Is it that 1) may not be a proposition?

2. Determine the truth value of the following statements.

a) If $1 + 1 = 2$, then $2 + 2 = 5$.

b) If $1 + 1 = 2$, then dogs can fly.

c) If $1 + 1 = 3$, then unicorns exist.

Solution: a) False, b) False, c) True.

3. State the converse, contrapositive, and inverse of each of these conditional statements.
a) I come to the class if there is going to be a quiz.
b) A positive integer is a prime only if it has no divisors other than 1 and itself.
Solution:
a) If there is going to be a quiz, then I will come to the class.
Converse: If I come to the class, then there will be a quiz.
Contrapositive: If I do not come to the class, then there will be no quiz.
Inverse: If there is no quiz, then I will not come to the class.
b) If a positive integer is a prime, then it has no divisors other than 1 and itself.
Converse: If a positive integer has no divisors other than 1 and itself, then it is a prime.
Contrapositive: If a positive integer has divisors other than 1 and itself, then it is not a prime.
Inverse: If a positive integer is not a prime, then it has divisors other than 1 and itself.

4. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

Solution: No, there cannot be such a barber. There is a contradiction.
If the barber shaves himself, then he cannot shave himself.
If the barber doesn't shave himself, then he should shave himself.

5. Construct a compound proposition that corresponds to exclusive OR.

Solution: $(\neg x \vee \neg y) \wedge (x \vee y)$.

6. An island has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?
Solution: Both are knaves. Analyse all possible cases. Only the case where both are knaves is consistent with their statements.

7. Determine whether the following statements are tautology, contradiction or contingency

a) $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$

b) $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

Solution:
a) Contingency. (Construct the truth table to find out.)
b) Tautology. (Construct the truth table to find out.)

8. Show that the following pairs are logically equivalent.

a) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

b) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

c) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \equiv T$

Solution:

a) $\neg p \rightarrow (q \rightarrow r)$

$\equiv \neg p \rightarrow (\neg q \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv \neg(\neg p) \vee (\neg q \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv p \vee (\neg q \vee r)$ (Double negation law)

$\equiv (p \vee \neg q) \vee r$ (Associative law)

$\equiv (\neg q \vee p) \vee r$ (Commutative law)

$\equiv \neg q \vee (p \vee r)$ (Associative law)

$\equiv q \rightarrow (p \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

b) $\neg(p \vee (\neg p \wedge q))$

$\equiv \neg((p \vee \neg p) \wedge (p \vee q))$ (Distributive law)

$\equiv \neg(T \wedge (p \vee q))$ (Negation law)

$\equiv \neg((p \vee q) \wedge T)$ (Commutative law) (Do not forget this step)

$\equiv \neg(p \vee q)$ (Identity law)

$\equiv \neg p \wedge \neg q$ (De Morgan's law)

c) Since the proof is very long, I will be skipping a few obvious steps or apply more than one rule of inference in one step.

$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

$\equiv ((\neg p \vee q) \wedge (\neg q \vee r)) \rightarrow (\neg p \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r)$ (Using $p \rightarrow q \equiv \neg p \vee q$)

$\equiv ((p \wedge \neg q) \vee (q \wedge \neg r)) \vee (\neg p \vee r)$ (De Morgan's law)

$\equiv (((p \wedge \neg q) \vee (q \wedge \neg r)) \vee \neg p) \vee r$ (Associative law)

$\equiv (\neg p \vee ((p \wedge \neg q) \vee (q \wedge \neg r))) \vee r$ (Commutative law)

$\equiv ((\neg p \vee (p \wedge \neg q)) \vee (q \wedge \neg r)) \vee r$ (Associative law)

$\equiv (((\neg p \vee p) \wedge (\neg p \vee \neg q)) \vee (q \wedge \neg r)) \vee r$ (Distributive law)

$\equiv ((T \wedge (\neg p \vee \neg q)) \vee (q \wedge \neg r)) \vee r$ (Negation law)

$\equiv ((\neg p \vee \neg q) \vee (q \wedge \neg r)) \vee r$ (Identity law)

$\equiv (\neg p \vee (\neg q \vee (q \wedge \neg r))) \vee r$ (Associative law)

$\equiv (\neg p \vee ((\neg q \vee q) \wedge (\neg q \vee \neg r))) \vee r$ (Distributive law)

$\equiv (\neg p \vee (T \wedge (\neg q \vee \neg r))) \vee r$ (Negation law)

$\equiv (\neg p \vee (\neg q \vee \neg r)) \vee r$ (Identity law)

$\equiv ((\neg p \vee \neg q) \vee \neg r) \vee r$ (Associative law)

$\equiv (\neg p \vee \neg q) \vee (\neg r \vee r)$ (Identity law)

$\equiv (\neg p \vee \neg q) \vee T$ (Negation law)

$\equiv T$ (Domination law)

9. Find the truth value of the following propositions where the domain is the set of positive integers. Justify your answer briefly.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x < 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Solution:

a) False. Counterexample, $x = 2$.

b) True. For every x, take y = 1.

c) False. Counterexample, $x = 1, y = 1$ or $x = 4, y = 8$.